Leptogenesis with "Fuzzy Mass Shell" for Majorana Neutrinos

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Abstract

We study the mixing of elementary and composite particles. In quantum field theory the mixing of composite particles originates in the couplings of the constituent quarks and for neutrinos in self–energy diagrams. In the event that the incoming and outgoing neutrinos have different masses, the self–energy diagrams vanish because energy is not conserved but the finite decaying widths make the mixing possible. We can consider the neutrinos to be "fuzzy" states on their mass shell and the mixing is understood as the overlap of two wavefunctions. These considerations restrict the mass difference to be approximately equal to or smaller than the largest of the two widths: $|M_i - M_j| \lesssim max\{\Gamma_i, \Gamma_j\}$.

1 Lagrangian with Majorana neutrinos.

During the last few years a lot of attention was paid to the possibility of creating a baryon asymmetry through leptogenesis. The proposed schemes introduce heavy Majorana neutrinos with CP–violating couplings, where both the so–called "direct" and "indirect" contributions to leptogenesis were considered. In calculating the lepton asymmetry, the authors consider the Standard Model (SM) with the usual particle content plus three Majorana neutrinos, which are singlets under the weak SU(2)–group [1] - [10].

The part of the Lagrangian with Majorana neutrinos consists of the Majorana mass term and the Yukawa interactions of these neutrinos with leptons and Higgs bosons:

$$L = \frac{1}{2} \sum_i M_i \overline{N_i} \, N_i + \sum_{\alpha, i} h_{\alpha i} \overline{l_{L\alpha}} \, \phi \, P_R N_i + \sum_{\alpha, i} h_{\alpha i}^* \overline{N_i} P_L \, l_{L\alpha} \phi^+ + h.c. \tag{1}$$

In this Lagrangian $l_{L\alpha}$ are left-handed lepton doublets of the SM, ϕ is the Higgs doublet of the SM, $\alpha, i = 1, 2, 3$ denote the index of fermion generation and N_i is the self-conjugate Majorana field:

$$N_i = N_{Ri} + (N_{Ri})^c = \begin{pmatrix} 0 \\ N_{Ri} \end{pmatrix} + \begin{pmatrix} (N_{Ri})^c \\ 0 \end{pmatrix} = \begin{pmatrix} (N_{Ri})^c \\ N_{Ri} \end{pmatrix}. \tag{2}$$

One should remember here that $(N_{Ri})^c$ is a left-handed antiparticle $(N_{Ri})^c = (N_i^c)_L$.

Another (more explicit) way of writing this Lagrangian is used in [7]:

$$L = \sum_i M_i \left[\overline{(N_{Ri})^c} N_{Ri} + \overline{N_{Ri}} (N_{Ri})^c \right] + \sum_{\alpha, i} h_{\alpha i} \overline{l_{L\alpha}} N_{Ri} \phi + \sum_{\alpha, i} h_{\alpha i}^* \overline{N_{Ri}} l_{L\alpha} \phi^+$$
$$+ \sum_{\alpha, i} h_{\alpha i} \overline{(N_{Ri})^c} (l_{L\alpha})^c \phi + \sum_{\alpha, i} h_{\alpha i}^* \overline{(l_{L\alpha})^c} (N_{Ri})^c \phi^+ . \tag{3}$$

One can easily show that the two Lagrangians (1) and (3) are identical. In fact, by the definition of charge–conjugate fields one can show that

$$\overline{(l_{L\alpha})^c} \phi^+ (N_{Ri})^c = -(l_{L\alpha})^T C^{-1} \phi^+ C \overline{N_{Ri}}^T = -(l_{L\alpha})^T \phi^+ \overline{N_{Ri}}^T = \overline{N_{Ri}} \phi^+ l_{L\alpha}. \tag{4}$$

Notice that the mass term in (1) violates lepton number by two units and the Yukawa interaction terms violate the CP-symmetry. In general, it is desirable to have a gauge theory with a symmetry responsible for lepton number conservation. In such a theory, the Majorana mass term is generated as the result of spontaneous breaking of lepton number. This theory can be the goal of future investigations; meanwhile, in the model under consideration, the asymmetry is believed to be generated at temperatures bigger than the electroweak symmetry scale, but lower than the scale where the Majorana mass is created. Thus the Lagrangian (1) is an "intermediate–energy" effective Lagrangian. At the high energies considered here, the vacuum expectation value for the Higgs condensate is very small, so that the masses (here we mean vacuum masses and neglect temperature contributions) of the charged leptons m_α and Higgs particles m_ϕ are negligibly small or zero.

2 Indirect contribution to leptogenesis.

The Lagrangian (1) was considered several times [1] - [5], where the so-called "direct" contribution to leptogenesis was computed. Here we are interested in the "indirect" contribution to leptogenesis, calculated in references [6]–[10] and reviewed recently in [11].
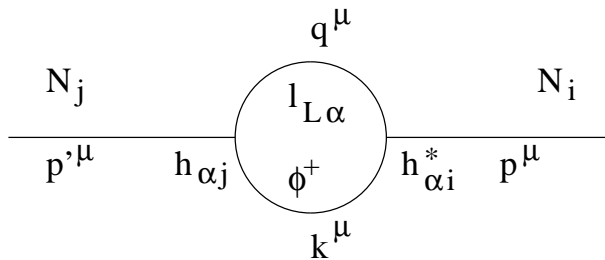
Figure 1: The self-energy diagram of the heavy Majorana neutrino

The indirect contribution is described by the self-energy diagram in Fig.1. This diagram was investigated in the context of two different approaches. In the first one the self–energy diagram is considered to be the *intermediate* state of some physical process [12, 13] (for example, lepton–Higgs scattering [14]). In this case the four-momentum of the Majorana neutrino is off the mass shell and is determined by the four-momenta of the incoming (outgoing) lepton and Higgs boson. We wish to emphasize is that there are no restrictions on the value of neutrino four-momentum. In this approach the lepton asymmetry appears in the scattering process $l^c \phi \to l \phi^+$, through interference of the tree-level diagram with the one-loop diagram [11].

The second approach, presented earlier in [6, 7], uses the notions of transition $N_j \to N_i$, originating from the one–loop diagram mentioned already. If $i = j$ then the diagram is analogous to any self-energy diagram and describes the correction to the fermion (Majorana neutrino in our case) mass. The precise role and calculation of this sort of diagrams is described in textbooks (see, for example, corrections to the electron mass in [15, 16].

A new situation appears for $i \neq j$. Now we have a free stable particle with mass M_i as incoming particle and it is impossible to transform it into a stable particle with mass M_j, because the process does not conserve energy and momentum. This transition can happen for unstable particles provided their mass difference is comparable to their widths. In addition the Majorana states are in a thermal bath interacting with the other fields so that energy and momentum is continuously exchanged with the background fields. If the transition probabilities are smooth functions of energies and widths, then it is a good approximation to calculate the transition amplitudes and probabilities per unit time and volume for unstable Majorana neutrinos and then introduce the results into the Boltzmann equations. In the latter step we also introduce the particle densities for initial and final states, as dictated by the thermodynamics of the early universe. A prototype of the thermodynamic calculation is described in ref. [14], [17] and recent articles [18, 19]. The interaction with the background fields is an integral part in the generation of the lepton asymmetry. In this article we calculate how the transition probabilities and the asymmetry are generated for unstable particles.

Before we address this topic, we review how the mixing occurs in a few known cases. The mixing of the K^0 with $\bar{K}^0$ is described by the box–diagram. In this case the K^0–mesons are treated as composite particles made up of quarks. The mixing diagrams are computed at the quark level, which are the fields of the basic Lagrangian (Standard Model). In the diagrams appear vertices of the form $\bar{u}\, s\, W$, because the quarks mix in the Lagrangian through the Cabibbo–Kobayashi–Maskawa matrix. Thus the mixings in the above case originate from the mixings at vertices between quarks of different charges and manifest themselves as the mixing of mesonic states through the interaction of their constituent quarks. In the above case energy and momen-

tum is conserved. For the mesonic states the mass difference is comparable to the widths. For K–mesons

$$|\Delta M_K| = \frac{1}{2}|\Delta \Gamma_K| \sim \frac{1}{2}\Gamma_K$$

and for the B–mesons [20]

$$\left(\frac{\Delta M}{\Gamma}\right)_{B_d} = 0.734 \pm 0.035\,.$$

The neutrinos have charged current couplings to the leptons where the lepton number is conserved to a large degree of accuracy. The Majorana couplings, on the other hand, may have lepton number violating charged couplings as seen in the Lagrangian of eqs. (1) or (3). The mass matrix of Majorana neutrinos in eq. (1) can be made real and at the classical (tree) level the generations do not mix. However, it was noted in ref. [7] that the one–loop effects can mix the Majorana states, i.e. the states defined at tree level are not the physical ones. To state the result in another way, we find that Majorana neutrinos of different generation mix, while the electron and the muon do not mix.

In this paper we show that the instability of the Majorana neutrinos is of principal importance for the mixing phenomenon. The neutrinos have finite widths and can be treated, through the uncertainty principle, as "fuzzy states" on their mass shell. This allows us to define the mixing of physical states in the case when the difference of the tree–level neutrino masses is less or about equal to their widths:

$$|M_i - M_j| \lesssim max\{\Gamma_i,\ \Gamma_j\}. \tag{5}$$

Condition (5) can evidently be applied to any other particles. For electron and muon, however, $m_\mu - m_e \gg \Gamma_\mu$ and consequently the mixing is very small.

3   S-matrix for the mixing of unstable particles.

The general S-matrix theory necessarily uses the notion of asymptotic states. This means that states of incoming, $|in(t \to -\infty)\rangle$, and outgoing, $|out(t \to +\infty)\rangle$, particles are defined as states of free (noninteracting) particles at infinite times. The second principal element of the $S-$matrix theory presupposes that interactions do not exist at all "infinite" times, but are "turned on adiabatically" from $t \to -\infty$ to $t = 0$ and also "turn off" adiabatically from $t = 0$ to $t \to +\infty$. One, of course, understands that "infinite" here means macroscopically large in comparison to the time of interaction. This general argument can naturally be applied to the case of unstable particles. In this case the time of "turning on" and "off" the interaction is easily estimated to be of the order of the inverse particle width. We try to express this in a phenomenological definition (the idea is similar to that of Breit and Wigner when considering the cross section at resonance) of field operators for unstable particles. For neutrinos we introduce

$$N_i(x) = \sum_{\vec{k},\lambda} \left[u^i(\vec{k},\lambda) a^i_{\vec{k},\lambda} e^{-i\sqrt{\vec{k}^2+M_i^2}\,t} e^{i\vec{k}\vec{x}} + v^i(\vec{k},\lambda) a^{i+}_{\vec{k},\lambda} e^{i\sqrt{\vec{k}^2+M_i^2}\,t} e^{-i\vec{k}\vec{x}} \right] e^{-\Gamma_i|t|} \tag{6}$$

The indices $i = 1, 2$ denote the generation of the neutrino and for the sake of simplicity we consider only two generations. $a^{i+}_{\vec{k},\lambda}$ and $a^i_{\vec{k},\lambda}$ are the fermionic creation and annihilation operators. Formula (6) reflects the fact that neutrinos "disappear" (decay) for infinite time ($N_i \to 0$ at $t \to \pm\infty$). We assume that the widths are generated by the Lagrangian in eq. (1) and calculated by the decay diagrams or from the absorptive part of the diagonal terms of the self–energy. Eq. (6) is the

only step which is not proved from field theory, since field theory does not consider asymptotic states. We think, however, that the physical meaning of eq. (6) is rather clear and will help us understand the mixing of Majorana neutrinos.

In this paper we will follow the idea of an earlier paper [7] and will introduce an "effective Hamiltonian" with non-diagonal terms in the mass matrix. Initially they are calculated as elements of the S-matrix to second order in the Yukawa–couplings by using perturbation theory. Let i and j be neutrinos of definite flavors, then

$$S_{ij} = -\frac{1}{2} \int d^4x \int d^4x' \langle i|L_{int}(x)L_{int}(x')|j\rangle, \tag{7}$$

with the neutrino fields defined by eq. (6).

4 Loop calculation with unstable Majorana neutrinos.

As mentioned in the previous section, we shall use perturbation theory. To second order in the Yukawa couplings the $S-$matrix element, corresponding to the one-loop diagram, is given by (7). ¿From the product of the interaction Lagrangians

$$\langle i| \left\{ h_{\alpha m}\overline{l_{L\alpha}}(x)\phi(x)P_R N_m(x) + h^*_{\alpha m}\overline{N_m}(x)P_L\phi^+(x)l_{L\alpha}(x) \right\} \times$$

$$\times \left\{ h_{\alpha n}\overline{l_{L\alpha}}(x')\phi(x')P_R N_n(x') + h^*_{\alpha n}\overline{N_n}(x')P_L\phi^+(x')l_{L\alpha}(x') \right\} |j\rangle$$

two terms contribute to (7):

$$2h^*_{\alpha i}h_{\alpha j} \cdot \overline{N_i}(x)P_L\phi^+(x)l_{L\alpha}(x) \cdot \overline{l_{L\alpha}}(x')\phi(x')P_R N_j(x')$$

$$+2h_{\alpha i}h^*_{\alpha j} \cdot P_R N_i(x)\phi(x)\overline{l_{L\alpha}}(x) \cdot \overline{N_j}(x')P_L\phi^+(x')l_{L\alpha}(x'). \tag{8}$$

The new aspect of our calculation is the form of the field operators for unstable neutrinos given in eq. (6). We calculate in detail the first term, using Wick's expansion for the product of operators

$$S_{ij} = \frac{1}{2} \int d^4x \int d^4x' 2h^*_{\alpha i}h_{\alpha j}\bar{u}^i_{p\sigma}P_L e^{i\sqrt{\vec{p}^2+M_i^2}t - \Gamma_i|t| - i\vec{p}\vec{x}} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{\not{q} - m_\alpha + i\varepsilon}$$

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \left(\frac{1}{k^2 - m_\phi^2 + i\varepsilon} - \frac{1}{k^2 - \Lambda^2 + i\varepsilon} \right) e^{-i\sqrt{\vec{p}'^2+M_j^2}t' - \Gamma_j|t'| + i\vec{p}'\vec{x}'} P_R u^j_{p'\sigma'}, \tag{9}$$

where Λ is a parameter for the Pauli–Villars regularization.[1] After integration over d^3x, d^3k, d^3x' one obtains:

$$S_{ij} = \frac{1}{2} \int dt \int dt' 2h^*_{\alpha i}h_{\alpha j}\bar{u}^i_{p\sigma}P_L \int \frac{d^4q}{(2\pi)^4} \frac{\not{q} + m_\alpha}{q^2 - m_\alpha^2 + i\varepsilon} \int \frac{dk^0}{2\pi} \frac{1}{k^{02} - (\vec{p}-\vec{q})^2 - m_\phi^2 + i\varepsilon} \tag{10}$$

$$e^{i(\sqrt{\vec{p}^2+M_i^2}-k^0-q^0)t - \Gamma_i|t|} e^{i(\sqrt{\vec{p}'^2+M_j^2}-k^0-q^0)t' - \Gamma_j|t'|} \delta(\vec{p}-\vec{p}')(2\pi)^3 P_R u^j_{p'\sigma'}.$$

with a similar expression for the terms with the Pauli-Villars regularization parameter. An important feature of this formula is the product of the two integrals

$$J_1 = \int dt\, e^{i(\sqrt{\vec{p}^2+M_i^2}-k^0-q^0)t - \Gamma_i|t|} \qquad J_2 = \int dt'\, e^{i(\sqrt{\vec{p}'^2+M_j^2}-k^0-q^0)t' - \Gamma_j|t'|}. \tag{11}$$

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[1]Another regularization is equally possible.

In the case of vanishing widths $\Gamma_i = \Gamma_j = 0$ each of the integrals is equal to a $\delta-$function and their product $\delta(\sqrt{\vec{p}^2 + M_i^2} - k^0 - q^0)\delta(\sqrt{\vec{p}^2 + M_j^2} - k^0 - q^0)$ is zero except when $M_i = M_j$ (with $\vec{p} = \vec{p}'$). This result is a mathematical demonstration of physical arguments, discussed in section 2. On the other hand, for non–zero widths the integrals J_1 and J_2 lead to "bell-shaped" Lorentzian functions:

$$J_1 \cdot J_2 = (2\pi)^2 \frac{\Gamma_i/\pi}{\Gamma_i^2 + (\sqrt{\vec{p}^2 + M_i^2} - k^0 - q^0)^2} \cdot \frac{\Gamma_j/\pi}{\Gamma_j^2 + (\sqrt{\vec{p}^2 + M_j^2} - k^0 - q^0)^2}. \tag{12}$$

This product is non–zero and large when condition (5) is satisfied; its non–zero value can be graphically understood as the "overlap" of two "bells". So the limit $|M_i - M_j| \gg \Gamma_{i(j)}$, which is often used in the calculation of asymmetry, leads to very small overlap functions. Another remark concerns the three-dimensional $\delta-$function with zero argument $(2\pi)^3\delta(\vec{p}-\vec{p}') = (2\pi)^3\delta(\vec{0})$, which appears in eq. (10). We will keep it throughout the section and it corresponds to the volume element where the interaction takes place.

When we substitute (12) into (10), and in addition integrate over k^0, we obtain the first order expression for $\Gamma_{i(j)}$:

$$S_{ij} = h_{\alpha i}^* h_{\alpha j} 2\pi\delta(\vec{p} - \vec{p})(2\pi)^3 \bar{u}_{p\sigma}^i P_L \int \frac{d^4q}{(2\pi)^4} \frac{\not{q} + m_\alpha}{q^2 - m_\alpha^2 + i\varepsilon} P_R u_{p\sigma'}^j$$

$$\left\{ \frac{\Gamma_j}{\pi} \frac{1}{(\sqrt{\vec{p}^2 + M_i^2} - \sqrt{\vec{p}^2 + M_j^2})^2 + \Gamma_j^2} \times \right.$$

$$\times \left(\frac{1}{(\sqrt{\vec{p}^2 + M_i^2} - q^0)^2 - (\vec{p} - \vec{q})^2 - m_\phi^2} - \frac{1}{(\sqrt{\vec{p}^2 + M_i^2} - q^0)^2 - (\vec{p} - \vec{q})^2 - \Lambda^2} \right) \tag{13}$$

$$+ \frac{\Gamma_i}{\pi} \frac{1}{(\sqrt{\vec{p}^2 + M_j^2} - \sqrt{\vec{p}^2 + M_i^2})^2 + \Gamma_i^2} \times$$

$$\left. \times \left(\frac{1}{(\sqrt{\vec{p}^2 + M_j^2} - q^0)^2 - (\vec{p} - \vec{q})^2 - m_\phi^2} - \frac{1}{(\sqrt{\vec{p}^2 + M_j^2} - q^0)^2 - (\vec{p} - \vec{q})^2 - \Lambda^2} \right) \right\}$$

This expression, as is easily seen, reproduces the usual expression for the loop integral after one introduces the zero-component of four momentum p^0:

$$S_{ij} = h_{\alpha i}^* h_{\alpha j} 2\pi\delta(\vec{p} - \vec{p})(2\pi)^3 \bar{u}_{p\sigma}^i P_L \int \frac{d^4q}{(2\pi)^4} \frac{\not{q} + m_\alpha}{q^2 - m_\alpha^2 + i\varepsilon} P_R u_{p\sigma'}^j \times$$

$$\times \left\{ \frac{\Gamma_j}{\pi} \frac{1}{(p^0 - \sqrt{\vec{p}^2 + M_j^2})^2 + \Gamma_j^2} \frac{1}{(p^0 - q^0)^2 - (\vec{p} - \vec{q})^2 - m_\phi^2} \Big|_{p^0 = \sqrt{\vec{p}^2 + M_i^2}} \right.$$

$$\left. + \frac{\Gamma_i}{\pi} \frac{1}{(p^0 - \sqrt{\vec{p}^2 + M_i^2})^2 + \Gamma_i^2} \frac{1}{(p^0 - q^0)^2 - (\vec{p} - \vec{q})^2 - m_\phi^2} \Big|_{p^0 = \sqrt{\vec{p}^2 + M_j^2}} \right\} \tag{14}$$

Integrals of this type are standard and the result for the limit $m_\alpha \to 0$, $m_\phi \to 0$ is well known:

$$Int = \lim_{m_\alpha, m_\phi \to 0} \int \frac{d^4q}{(2\pi)^4} \frac{\not{q} + m_\alpha}{q^2 - m_\alpha^2 + i\varepsilon} \left(\frac{1}{(p^\mu - q^\mu)^2 - m_\phi^2 + i\varepsilon} - \frac{1}{(p^\mu - q^\mu)^2 - \Lambda^2 + i\varepsilon} \right) =$$

$$= -i\gamma^\mu p_\mu \left(g_{dis} - \frac{i}{2}g_{abs} \right), \qquad g_{dis} = -\frac{1}{16\pi^2} \left(\frac{1}{2} \ln \frac{\Lambda^2}{p^2} + \frac{3}{4} \right), \qquad g_{abs} = \frac{1}{16\pi}$$

It is recognized that the dispersive part of this integral (g_{dis}) can be reabsorbed in the definition of coupling constants, while only the absorptive part (g_{abs}) survives and has physical consequences; for more details see [15, 16] and references therein.

Now the matrix element S_{ij} is given by

$$S_{ij} = h_{\alpha i}^* h_{\alpha j} 2\pi \delta(\vec{p} - \vec{p})(2\pi)^3(-i) \times$$

$$\times \left\{ \bar{u}_{p\sigma}^i P_L \left[\gamma^0 (\sqrt{\vec{p}^2 + M_i^2}) - \vec{\gamma}\vec{p} \right] \frac{\Gamma_j}{\pi} \frac{-i/2 \cdot g_{abs}}{(\sqrt{\vec{p}^2 + M_i^2} - \sqrt{\vec{p}^2 + M_j^2})^2 + \Gamma_j^2} P_R u_{p\sigma'}^j \right. \tag{15}$$

$$\left. + \bar{u}_{p\sigma}^i P_L \left[\gamma^0 (\sqrt{\vec{p}^2 + M_j^2}) - \vec{\gamma}\vec{p} \right] \frac{\Gamma_i}{\pi} \frac{-i/2 \cdot g_{abs}}{(\sqrt{\vec{p}^2 + M_j^2} - \sqrt{\vec{p}^2 + M_i^2})^2 + \Gamma_i^2} P_R u_{p\sigma'}^j \right\}$$

We transform this expression with the help of the Dirac equation

$$\bar{u}_{p\sigma}^i P_L \left[\gamma^0 \sqrt{\vec{p}^2 + M_i^2} - \vec{\gamma}\vec{p} \right] = \bar{u}_{p\sigma}^i P_R M_i,$$

$$\left[\gamma^0 \sqrt{\vec{p}^2 + M_j^2} - \vec{\gamma}\vec{p} \right] P_R u_{p\sigma'}^j = M_j P_L u_{p\sigma'}^j,$$

$$\bar{v}_{p\sigma'}^j P_L \left[\gamma^0 \sqrt{\vec{p}^2 + M_j^2} - \vec{\gamma}\vec{p} \right] = -\bar{v}_{p\sigma'}^j P_R M_j, \tag{16}$$

$$\left[\gamma^0 \sqrt{\vec{p}^2 + M_i^2} - \vec{\gamma}\vec{p} \right] P_R v_{p\sigma}^i = -M_i P_L v_{p\sigma}^i.$$

Finally we arrive at

$$S_{ij}^{(I)} = h_{\alpha i}^* h_{\alpha j} 2\pi \delta(\vec{p} - \vec{p})(2\pi)^3(-i)\frac{1}{2}\bar{u}_{p\sigma}^i \left\{ M_i \frac{\Gamma_j}{\pi} \frac{-i/2 \cdot g_{abs}}{(\sqrt{\vec{p}^2 + M_i^2} - \sqrt{\vec{p}^2 + M_j^2})^2 + \Gamma_j^2} \right.$$

$$\left. + M_j \frac{\Gamma_i}{\pi} \frac{-i/2 \cdot g_{abs}}{(\sqrt{\vec{p}^2 + M_j^2} - \sqrt{\vec{p}^2 + M_i^2})^2 + \Gamma_i^2} \right\} u_{p\sigma'}^j \tag{17}$$

This is the final expression for the S–matrix element originating from the first term in eq. (8). The second term from eq. (8) leads us to the similar expression with v−spinors

$$S_{ij}^{(II)} = h_{\alpha i} h_{\alpha j}^* 2\pi \delta(\vec{p} - \vec{p})(2\pi)^3(-i)\frac{1}{2}\bar{v}_{p\sigma'}^j \left\{ M_i \frac{\Gamma_j}{\pi} \frac{-i/2 \cdot g_{abs}}{(\sqrt{\vec{p}^2 + M_i^2} - \sqrt{\vec{p}^2 + M_j^2})^2 + \Gamma_j^2} \right.$$

$$\left. + M_j \frac{\Gamma_i}{\pi} \frac{-i/2 \cdot g_{abs}}{(\sqrt{\vec{p}^2 + M_j^2} - \sqrt{\vec{p}^2 + M_i^2})^2 + \Gamma_i^2} \right\} v_{p\sigma}^i \tag{18}$$

As mentioned already, $\delta(\vec{p} - \vec{p})(2\pi)^3$ represents the volume element V. Similarly the time of interaction also appears as multiplicative factor.

$$J_{ij} = \int_{-\infty}^{\infty} dt\, e^{-i\sqrt{\vec{p}^2 + M_i^2}t - \Gamma_i|t|} e^{i\sqrt{\vec{p}^2 + M_j^2}t - \Gamma_j|t|} = 2\pi \frac{(\Gamma_i + \Gamma_j)/\pi}{(\Gamma_i + \Gamma_j)^2 + (\sqrt{\vec{p}^2 + M_i^2} - \sqrt{\vec{p}^2 + M_j^2})^2}. \tag{19}$$

The new feature of this calculation is the presence of the masses and the decay widths in the S–matrix. The new terms define the time intervals when the interaction takes place. To obtain the transition amplitudes per unit volume and unit time we must divide by the factor $V \cdot J_{ij}$ where V is the volume element and J_{ij} the time interval of the interaction. In other words, we shall work with the "T–Matrix" defined through the equation

$$S_{fi} = 1 + i \, T_{fi}(2\pi)^4 \delta^4(p_f - p_i) \tag{20}$$

where the energy conserving δ–function will be substituted by the expression

$$(2\pi)\delta(E_f - E_i) \rightarrow \frac{2(\Gamma_i + \Gamma_j)}{(E_f - E_i)^2 + (\Gamma_f + \Gamma_i)^2} \,. \tag{21}$$

One easily arrives at

$$T_{ij} = \frac{S_{ij}^{(I)} + S_{ij}^{(II)}}{V \cdot J_{ij}} = \frac{(-1)}{2} \frac{1}{16\pi} \left(h_{\alpha i}^* h_{\alpha j} \bar{u}_{p\sigma}^i \left[M_i \frac{\Gamma_j}{\Gamma_i + \Gamma_j} P_R + M_j \frac{\Gamma_i}{\Gamma_i + \Gamma_j} P_L \right] u_{p\sigma'}^j \right.$$
$$\left. + h_{\alpha i} h_{\alpha j}^* \bar{v}_{p\sigma'}^j \left[M_j \frac{\Gamma_i}{\Gamma_i + \Gamma_j} P_R + M_i \frac{\Gamma_j}{\Gamma_i + \Gamma_j} P_L \right] v_{p\sigma}^i \right). \tag{22}$$

Notice the S–matrix element (17), (18) vanish in the case $|M_i - M_j| \gg \Gamma_{ij}$ since the Lorentzian representation of the Delta–function (21) tends to zero. It is easy to see that the same occurs for the transition probability

$$\frac{w(N_i \rightarrow N_j)}{V \cdot J_{ij}} = |T_{ij}|^2 \cdot (2\pi)^4 \cdot \delta(\vec{p} - \vec{p}') \cdot \frac{(\Gamma_i + \Gamma_j)/\pi}{(E_f - E_i)^2 + (\Gamma_f + \Gamma_i)^2} \,.$$

Let us summarize our results. We have shown that neutrinos of different generations can mix through one–loop diagrams even if they have different masses. The physical reason of this possibility is related to the final widths of neutrinos : their mass shells are "fuzzy" and energy is conserved with the accuracy comparable to their width. The closer the masses of neutrinos, the larger is the transition $N_i \rightarrow N_j$ probability; and for $|M_i - M_j| \gg \Gamma_{i(j)}$ the mixing has no physical meaning.

As we already know, neutrino mixing provides an indirect contribution to leptogenesis. As it was previously done in [6, 7], we are working in terms of an effective Hamiltonian, or, what is the same, in terms of an effective mass matrix. From the transition matrix elements and the mass term in eq. (1) we can now proceed to calculate an effective mass matrix and the physical states.

We consider the expression for T_{ij} as the first order contribution for mass matrix elements. One should notice that the contributions are different for R and L parts of our 4–spinor N_i in eq. (2), i.e. they are different for N_{Ri} and $(N_{Ri})^c$. Evidently, an overall factor of $(-i)$ and the spinors originate from the definition of the S–matrix and do not appear on the mass matrix. A factor of $1/2$ is also omitted in order to be consistent with the Lagrangian in eq. (1), which has $1/2$ in front of the mass term.

So the corrections to the $M_i \overline{(N_{Ri})^c} N_{Ri}$ mass term (see Lagrangian in the form (3)) are

$$H_{ij} = \frac{(-i)}{2} \frac{1}{8\pi} \left(h_{\alpha i}^* h_{\alpha j} \frac{M_i \Gamma_j}{\Gamma_i + \Gamma_j} + h_{\alpha i} h_{\alpha j}^* \frac{M_j \Gamma_i}{\Gamma_i + \Gamma_j} \right)$$

and the corrections to the $M_i\,\overline{N_{Ri}}\,(N_{Ri})^c$ mass term are

$$\tilde{H}_{ij} = \frac{(-i)}{2}\,\frac{1}{8\pi}\left(h_{\alpha i}^*\,h_{\alpha j}\,\frac{M_j\Gamma_i}{\Gamma_i+\Gamma_j} + h_{\alpha i}\,h_{\alpha j}^*\,\frac{M_i\Gamma_j}{\Gamma_i+\Gamma_j}\right)$$

When $\Gamma_i = \Gamma_j$, this result coincides with those obtained in ref. [7]. Our account of finite widths of Majorana neutrinos slightly corrects the result which is not very essential. What is important, is that the definition in eq. (6) enables one to give physical meaning to the mixing of particles with different masses.

5 Effective Contribution to the Mass Term and Physical Neutrino States.

We shall work in a vector space with four basis vectors $(N_{R1}^c,\ N_{R1},\ N_{R2}^c,\ N_{R2})$ which are the states occurring in the interaction term of eq. (1). The transitions among these states introduce an effective matrix

$$m = \begin{pmatrix} 0 & M_1 + H_{11} & 0 & H_{12} \\ M_1 + H_{11} & 0 & \tilde{H}_{12} & 0 \\ 0 & H_{12} & 0 & M_2 + H_{22} \\ \tilde{H}_{12} & 0 & M_2 + H_{22} & 0 \end{pmatrix} \tag{23}$$

with

$$H_{ij} = 2\left[h_{\alpha i}^* h_{\alpha j}\frac{M_i\Gamma_j}{\Gamma_i+\Gamma_j} + h_{\alpha i}h_{\alpha j}^*\frac{M_j\Gamma_i}{\Gamma_i+\Gamma_j}\right]\left(-\frac{i}{2}\,g_{ab}\right) \tag{24}$$

$$\tilde{H}_{ij} = -H_{ij}^* \qquad \text{and} \qquad g_{ab} = \frac{1}{16\pi} \tag{25}$$

The 2×2 matrices in the upper left and lower right corners of eq. (23) are lepton–number violating but flavor conserving, while the 2×2 matrices along the off–diagonal are lepton– and flavor–number violating. All terms should be present and we found no way to reduce it to a 2×2 matrix. It remains to diagonalize the matrix and find the wavefunctions. It is instructive to present a perturbative solution of the problem and then discuss the exact solution.

We split the mass matrix into the dominant term and a perturbation

$$m = H_0 + \lambda V$$

with

$$H_0 = \begin{pmatrix} 0 & M_1 + H_{11} & 0 & 0 \\ M_1 + H_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 + H_{22} \\ 0 & 0 & M_2 + H_{22} & 0 \end{pmatrix} \tag{26}$$

and

$$\lambda V = \begin{pmatrix} 0 & 0 & 0 & \lambda H_{12} \\ 0 & 0 & -\lambda H_{12}^* & 0 \\ 0 & \lambda H_{12} & 0 & 0 \\ -\lambda H_{12}^* & 0 & 0 & 0 \end{pmatrix} \tag{27}$$

8

We introduce a small parameter λ in order to keep track of the perturbative corrections and at the very end set $\lambda = 1$. Applying perturbation theory we find the eigenfunctions to $0(\lambda^2)$

$$U_1 = \begin{pmatrix} 1 \\ 1 \\ X \\ Y \end{pmatrix} \qquad \text{and} \qquad U_2 = \begin{pmatrix} X' \\ Y' \\ 1 \\ 1 \end{pmatrix} \qquad (28)$$

with the eigenvalues $\lambda_1 = M_1 - i\frac{\Gamma_{11}}{2}$, $\lambda_2 = M_2 - i\frac{\Gamma_{22}}{2}$, and $H_{ii} = -\frac{i}{2}\Gamma_{ii}$. They have the time dependence $U_i\, e^{-i(M_i - i\frac{\Gamma_{ii}}{2})t}$, with $i = 1$ and 2 which is consistent with the time dependence introduced in eq. (6). The physical states are

$$\psi_1 = \frac{1}{\sqrt{N}}\left[|N_{R1}\rangle + |N_{R1}^c\rangle + X\,|N_{R2}^c\rangle + Y\,|N_{R2}\rangle\right] \qquad (29)$$

$$\psi_2 = \frac{1}{\sqrt{2}}\left[|N_{R2}\rangle + |N_{R2}^c\rangle + X'\,|N_{R1}^c\rangle + Y'\,|N_{R1}\rangle\right] \qquad (30)$$

with $X = \frac{H_{12}M_1 + H_{12}^*M_2}{M_1^2 - M_2^2}$, $Y = \frac{H_{12}M_2 + H_{12}^*M_1}{M_1^2 - M_2^2}$ and similar formulas for X' and Y'.

The results are the same as in ref. [5, 6] with an additional dependence on the widths. The interesting result is that the definition in eq. (6) makes it possible to give a physical meaning to the mixing of particles with different masses.

We can finally calculate the asymmetry for the decay of each state. The width for the decay of ψ_1 into leptons is

$$\Gamma(\psi_1 \to \ell + \ldots) \propto \sum_\alpha |h_{\alpha 1} + h_{\alpha 2}\,Y\,|^2 \qquad (31)$$

and into antileptons

$$\Gamma(\psi_1 \to \bar{\ell} + \ldots) \propto \sum_\alpha |h_{\alpha 1}^* + h_{\alpha 2}^*\,X\,|^2\,. \qquad (32)$$

The lepton asymmetry, defined as

$$\delta_1 = \frac{\Gamma_{\Psi_1 \to l\phi^+} - \Gamma_{\Psi_1 \to l^c\phi}}{\Gamma_{\Psi_1 \to l\phi^+} + \Gamma_{\Psi_1 \to l^c\phi}}, \qquad (33)$$

and it is straightforward to calculate it

$$\delta_1 = \frac{1}{8\pi}\frac{M_1\,M_2}{M_2^2 - M_1^2}\frac{Im\,(h_{\alpha 1}^*\,h_{\alpha 2})^2}{|h_{\alpha 1}|^2 + |h_{\alpha 2}\,X|^2 + Re\,h_{\alpha 1}^*h_{\alpha 2}(Y + X^*)}\,. \qquad (34)$$

In the above calculation we considered the case $|M_2 - M_1| \gg H_{12}$. In the case that the two Majorana neutrinos are nearly degenerate we must diagonalize exactly the matrix in eq. (23) and then we recover the resonance phenomenon introduced in eq. (17) of ref. [7].

Our study so far considered the neutrinos as free particles. In reality, however, they are in a backgound of fields interacting many times with the other particles. These interactions at a finite temperature can modify their masses and widths [18, 19]. In the present analysis we have found that within the range of validity of eqs. (18) and (19) the T–matrix in eq. (22) is a smooth function of masses and the widths, and for this reason we can introduce the matrix elements in the Boltzmann equations in order to study the development of the asymmetry. This approach was followed in ref. [14].

Summary

We have shown that finite widths of Majorana neutrinos play a principal role in producing neutrino mixing via self–energy diagrams. In our approach the widths of neutrinos are treated as "fuzzy" states on their mass shell and the mixing is understood as an "overlap" of two fuzzy mass shell states. For the mathematical realization of these ideas we changed phenomenologically the definition of asymptotic neutrino states, including their widths as in eq. (6). Concerning the results, we showed that these changes do not influence significantly the expressions for the effective mass matrix and for the asymmetry generated in the case of small mass difference. For large mass differences the asymmetry is suppressed.

Acknowledgement

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